

TRACE FORMULAE FOR SCHRÖDINGER OPERATORS WITH COMPLEX-VALUED POTENTIALS ON CUBIC LATTICES

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ABSTRACT. We consider Schrödinger operators with complex decaying potentials on the lattice. Using some classical results from Complex Analysis we obtain some trace formulae and using them estimate globally all zeros of the Fredholm determinant in terms of the potential.

1. INTRODUCTION

Let us consider the Schrödinger operator H acting in $\ell^2(\mathbb{Z}^d)$, $d \geq 3$ and given by

$$H = H_0 + V, \quad H_0 = \Delta,$$

where Δ is the discrete Laplacian on \mathbb{Z}^d given by

$$(\Delta f)(n) = \frac{1}{2} \sum_{j=1}^d (f(n + e_j) + f(n - e_j)), \quad n = (n_j)_{j=1}^d \in \mathbb{Z}^d,$$

for $f = (f_n)_{n \in \mathbb{Z}^d} \in \ell^2(\mathbb{Z}^d)$. Here $e_1 = (1, 0, \dots, 0), \dots, e_d = (0, \dots, 0, 1)$ is the standard basis of \mathbb{Z}^d . The operator $V = (V_n)_{n \in \mathbb{Z}^d}$, $V_n \in \mathbb{C}$, is a complex potential given by

$$(Vf)(n) = V_n f_n, \quad n \in \mathbb{Z}^d.$$

We assume that the potential V satisfies the following condition:

$$V \in \ell^{2/3}(\mathbb{Z}^d). \quad (1.1)$$

Note that the condition (1.1) implies that V can be factorised as

$$V = V_1 V_2, \quad \text{where } V_1 \in \ell^1(\mathbb{Z}^d), V_2 \in \ell^2(\mathbb{Z}^d), \quad (1.2)$$

with $V_1 = |V|^{2/3-1}V$ and $V_2 = |V|^{1/3}$.

Here $\ell^q(\mathbb{Z}^d)$, $q > 0$ is the space of sequences $f = (f_n)_{n \in \mathbb{Z}^d}$ such that $\|f\|_q < \infty$, where

$$\|f\|_q = \|f\|_{\ell^q(\mathbb{Z}^d)} = \begin{cases} \sup_{n \in \mathbb{Z}^d} |f_n|, & q = \infty, \\ \left(\sum_{n \in \mathbb{Z}^d} |f_n|^q \right)^{\frac{1}{q}}, & q \in (0, \infty). \end{cases}$$

Note that $\ell^q(\mathbb{Z}^d)$, $q \geq 1$ is the Banach space equipped with the norm $\|\cdot\|_q$. It is well-known that the spectrum of the Laplacian Δ is absolutely continuous and equals

$$\sigma(\Delta) = \sigma_{\text{ac}}(\Delta) = [-d, d].$$

It is also well known that if V satisfies (1.1), the essential spectrum of the Schrödinger operator H on $\ell^2(\mathbb{Z}^d)$ is

$$\sigma_{\text{ess}}(H) = [-d, d].$$

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However, this condition does not exclude appearance of the singular continuous spectrum on the interval $[-d, d]$. Our main goal is to find new trace formulae for the operator H with complex potentials V and to use these formulae for some estimates of complex eigenvalues in terms of potentials.

Note that some of the results obtained in this paper are new even in the case of real-valued potentials due to the presence of the measure σ (see Theorem 2.3) appearing in the canonical factorisation of the respective Fredholm determinants. Non-triviality of such a measure is due to the weak condition (1.1) on the potential V . We believe that it would be interesting to study the relation between properties of V and σ .

Recently uniform bounds on eigenvalues of Schrödinger operators in \mathbb{R}^d with complex-valued potentials decaying at infinity attracted attention of many specialists. We refer to [6] for a review of the state of the art of non-selfadjoint Schrödinger operators and for motivations and applications. Bounds on single eigenvalues were proved, for instance, in [1, 7, 16, 11] and bounds on sums of powers of eigenvalues were found in [13, 25, 8, 9, 4, 15, 12]. The latter bounds generalise the Lieb–Thirring bounds [26] to the non-selfadjoint setting. Note that in [15] (Theorem 16) the authors obtained estimates on the sum of the distances between the complex eigenvalues and the continuous spectrum $[0, \infty)$ in terms of L^p -norms of the potentials. Note that almost no results are known on the number of eigenvalues of Schrödinger operators with complex potentials. We refer here to a recent paper [14] where the authors discussed this problem in details in odd dimensions.

For the discrete Schrödinger operators most of the results were obtained in the self-adjoint case, see, for example, [36] (for the \mathbb{Z}^1 case). Schrödinger operators with decreasing potentials on the lattice \mathbb{Z}^d have been considered by Boutet de Monvel-Sahbani [5], Isozaki-Korotyaev [20], Kopylova [23], Rosenblum-Solomjak [29], Shaban-Vainberg [34] and see references therein. Ando [2] studied the inverse spectral theory for the discrete Schrödinger operators with finitely supported potentials on the hexagonal lattice. Scattering on periodic metric graphs \mathbb{Z}^d was considered by Korotyaev-Saburova [24].

Isozaki and Morioka (see Theorem 2.1. in [21]) proved that if the potential V is real and compactly supported, then the point-spectrum of H on the interval $(-d, d)$ is absent. Note that in [10] the author gave an example of embedded eigenvalue at the endpoint $\{\pm d\}$.

In this paper we use classical results from Complex Analysis that lead us to a new class of trace formula for the spectrum of discrete multi-dimensional Schrödinger operators with complex-valued potentials. In particular, we consider a so-called canonical factorisation of analytic functions from Hardy spaces via its inner and outer factors, see Section 6. Such factorisations allied for Fredholm determinants allow us to obtain trace formula that lead to some inequalities on the complex spectrum in terms of the $L^{2/3}$ norm of the potential function. Note also that in the case $d = 3$ we use a delicate uniform inequality for Bessel's functions obtained in Lemma 7.1.

2. SOME NOTATIONS AND STATEMENTS OF MAIN RESULTS

We denote by $\mathbb{D}_r(z_0) \subset \mathbb{C}$ the disc with radius $r > 0$ and center $z_0 \in \mathbb{C}$

$$\mathbb{D}_r(z_0) = \{z \in \mathbb{C} : |z - z_0| < r\},$$

and abbreviate $\mathbb{D}_r = \mathbb{D}_r(0)$ and $\mathbb{D} = \mathbb{D}_1$. Let also $\mathbb{T} = \partial\mathbb{D}$. It is convenient to introduce a new spectral variable $z \in \mathbb{D}$ by

$$\lambda = \lambda(z) = \frac{d}{2} \left(z + \frac{1}{z} \right) \in \Lambda = \mathbb{C} \setminus [-d, d], \quad z \in \mathbb{D}. \quad (2.3)$$

The function $\lambda(z)$ has the following properties:

- $z \rightarrow \lambda(z)$ is a conformal mapping from \mathbb{D} onto the spectral domain Λ .
- $\lambda(\mathbb{D}) = \Lambda = \mathbb{C} \setminus [-d, d]$ and $\lambda(\mathbb{D} \cap \mathbb{C}_{\mp}) = \mathbb{C}_{\pm}$.
- Λ is the cut domain with the cut $[-d, d]$, having the upper side $[-d, d] + i0$ and the lower side $[-d, d] - i0$. $\lambda(z)$ maps the boundary: the upper semi-circle onto the lower side $[-d, d] - i0$ and the lower semi-circle onto the upper side $[-d, d] + i0$.
- $\lambda(z)$ maps $z = 0$ to $\lambda = \infty$.
- The inverse mapping $z(\cdot) : \Lambda \rightarrow \mathbb{D}$ is given by

$$z = \frac{1}{d} \left(\lambda - \sqrt{\lambda^2 - d^2} \right), \quad \lambda \in \Lambda,$$

$$z = \frac{d}{2\lambda} + \frac{O(1)}{\lambda^3}, \quad \text{as } |\lambda| \rightarrow \infty.$$

Next we introduce the Hardy space $\mathcal{H}_p = \mathcal{H}_p(\mathbb{D})$. Let F be analytic in \mathbb{D} . For $0 < p \leq \infty$ we say F belongs to the Hardy space \mathcal{H}_p if F satisfies $\|F\|_{\mathcal{H}_p} < \infty$, where $\|F\|_{\mathcal{H}_p}$ is given by

$$\|F\|_{\mathcal{H}_p} = \begin{cases} \sup_{r \in (0,1)} \left(\frac{1}{2\pi} \int_{\mathbb{T}} |F(re^{i\vartheta})|^p d\vartheta \right)^{\frac{1}{p}}, & \text{if } 0 < p < \infty, \\ \sup_{z \in \mathbb{D}} |F(z)|, & \text{if } p = \infty. \end{cases}$$

Let \mathcal{B} denote the class of bounded operators and \mathcal{B}_1 and \mathcal{B}_2 be the trace and the Hilbert-Schmidt class equipped with the norm $\|\cdot\|_{\mathcal{B}_1}$ and $\|\cdot\|_{\mathcal{B}_2}$ respectively.

Denote by $D(z), z \in \mathbb{D}$ the determinant

$$D(z) = \det(I + V R_0(\lambda(z))), \quad z \in \mathbb{D},$$

where $R_0(\lambda) = (H_0 - \lambda)^{-1}, \lambda \in \Lambda$. The determinant $D(z), z \in \mathbb{D}$, is well defined for $V \in \mathcal{B}_1$ and if $\lambda_0 \in \Lambda$ is an eigenvalue of H , then $z_0 = z(\lambda_0) \in \mathbb{D}$ is a zero of D with the same multiplicity.

Theorem 2.1. *Let a potential V satisfy (1.1). Then the determinant $D(z) = \det(I + V R_0(\lambda(z))), z \in \mathbb{D}$, is analytic in \mathbb{D} . It has $N \leq \infty$ zeros $\{z_j\}_{j=1}^N$, such that*

$$0 < r_0 = |z_1| \leq |z_2| \leq \dots \leq |z_j| \leq |z_{j+1}| \leq |z_{j+2}| \leq \dots, \quad (2.4)$$

where $r_0 = \inf |z_j| > 0$.

Moreover, it satisfies

$$\|D\|_{\mathcal{H}_{\infty}} \leq e^{C\|V\|_{2/3}}, \quad (2.5)$$

where the constant C depends only on d .

Furthermore, the function $\log D(z)$ whose branch is defined by $\log D(0) = 0$, is analytic in the disk \mathbb{D}_{r_0} with the radius $r_0 > 0$ defined by (2.4) and it has the Taylor series as $|z| < r_0$:

$$\log D(z) = -c_1 z - c_2 z^2 - c_3 z^3 - c_4 z^4 - \dots \quad (2.6)$$

where

$$c_1 = d_1 a, \quad c_2 = d_2 a^2, \quad c_3 = d_3 a^3 - c_1, \quad c_4 = d_4 a^4 - c_2, \dots \quad (2.7)$$

$$d_1 = \operatorname{Tr} V, \quad d_2 = \operatorname{Tr} V^2, \quad d_3 = \operatorname{Tr} (V^3 + (3d/2)V), \dots, \quad (2.8)$$

and where $a = \frac{2}{d}$.

Define the Blaschke product $B(z)$, $z \in \mathbb{D}$ by

$$B(z) = \prod_{j=1}^N \frac{|z_j|}{z_j} \frac{(z_j - z)}{(1 - \bar{z}_j z)}, \quad \text{if } N \geq 1, \quad (2.9)$$

$$B = 1, \quad \text{if } N = 0.$$

Theorem 2.2. *Let a potential V satisfy (1.1) and let $N \geq 1$. Then the zeros $\{z_j\}$ of D in the disk \mathbb{D} labeled by (2.4) satisfy*

$$\sum_{j=1}^N (1 - |z_j|) < \infty. \quad (2.10)$$

Moreover, the Blaschke product $B(z)$, $z \in \mathbb{D}$ given by (2.9) converges absolutely for $\{|z| < 1\}$ and satisfies

i) $B \in \mathcal{H}_\infty$ with $\|B\|_{\mathcal{H}_\infty} \leq 1$,

$$\lim_{r \rightarrow 1} |B(re^{i\vartheta})| = |B(e^{i\vartheta})| = 1 \quad \text{for almost all } \vartheta \in \mathbb{T}, \quad (2.11)$$

and

$$\lim_{r \rightarrow 1} \int_0^{2\pi} \log |B(re^{i\vartheta})| d\vartheta = 0. \quad (2.12)$$

ii) The determinant D has the factorization in the disc \mathbb{D} :

$$D = BD_B,$$

where D_B is analytic in the unit disc \mathbb{D} and has not zeros in \mathbb{D} .

iii) The Blaschke product B has the Taylor series at $z = 0$:

$$\log B(z) = B_0 - B_1 z - B_2 z^2 - \dots \quad \text{as } z \rightarrow 0, \quad (2.13)$$

where B_n satisfy

$$B_0 = \log B(0) < 0, \quad B_1 = \sum_{j=1}^N \left(\frac{1}{z_j} - \bar{z}_j \right), \dots, \quad B_n = \frac{1}{n} \sum_{j=1}^N \left(\frac{1}{z_j^n} - \bar{z}_j^n \right), \dots$$

$$|B_n| \leq \frac{2}{r_0^n} \sum_{j=1}^N (1 - |z_j|).$$

The next statement describes the canonical representation of the determinant $D(z)$.

Theorem 2.3. *Let a potential V satisfy (1.1). Then*

i) There exists a singular measure $\sigma \geq 0$ on $[-\pi, \pi]$, such that the determinant D has a canonical factorization for all $|z| < 1$ given by

$$\begin{aligned} D(z) &= B(z)e^{-K_\sigma(z)}e^{K_D(z)}, \\ K_\sigma(z) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} d\sigma(t), \\ K_D(z) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} \log |D(e^{it})| dt, \end{aligned} \quad (2.14)$$

where $\log |D(e^{it})| \in L^1(-\pi, \pi)$.

ii) The measure σ satisfies

$$\text{supp } \sigma \subset \{t \in [-\pi, \pi] : D(e^{it}) = 0\}. \quad (2.15)$$

Remarks.

- 1) For the canonical factorisation of analytic functions see, for example, [22].
- 2) Note that for $D_{in}(z)$ defined by $D_{in}(z) = B(z)e^{-K_\sigma(z)}$, we have $|D_{in}(z)| \leq 1$, since $d\sigma \geq 0$ and $\text{Re} \frac{e^{it} + z}{e^{it} - z} \geq 0$ for all $(t, z) \in \mathbb{T} \times \mathbb{D}$.
- 3) The closure of the set $\{z_j\} \cup \text{supp } \sigma$ is called the spectrum of the inner function D_{in} .
- 4) $D_B = \frac{D}{B}$ has no zeros in the disk \mathbb{D} and satisfies

$$\log D_B(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} d\mu(t),$$

where the measure μ equals

$$d\mu(t) = \log |D(e^{it})| dt - d\sigma(t).$$

Theorem 2.4. (Trace formulae.) *Let V satisfy (1.1). Then the following identities hold*

$$\frac{\sigma(\mathbb{T})}{2\pi} - B_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |D(e^{it})| dt \geq 0, \quad (2.16)$$

$$-D_n + B_n = \frac{1}{\pi} \int_{\mathbb{T}} e^{-int} d\mu(t), \quad n = 1, 2, \dots \quad (2.17)$$

where $B_0 = \log B(0) = \log \left(\prod_{j=1}^N |z_j| \right) < 0$ and B_n are given by (2.13). In particular,

$$\sum_{j=1}^N \left(\frac{1}{z_j} - \bar{z}_j \right) = \frac{2}{d} \text{Tr } V + \frac{1}{\pi} \int_{\mathbb{T}} e^{-it} d\mu(t), \quad (2.18)$$

$$\sum_{j=1}^N \left(\frac{1}{z_j^2} - \bar{z}_j^2 \right) = \frac{4}{d^2} \text{Tr } V^2 + \frac{1}{\pi} \int_{\mathbb{T}} e^{-i2t} d\mu(t), \quad (2.19)$$

and

$$\begin{aligned} \sum_{j=1}^N \operatorname{Im} \lambda_j &= \operatorname{Tr} \operatorname{Im} V - \frac{d}{2\pi} \int_{\mathbb{T}} \sin t \, d\mu(t), \\ \sum_{j=1}^N \operatorname{Re} \sqrt{\lambda_j^2 - d^2} &= \operatorname{Tr} \operatorname{Re} V + \frac{d}{2\pi} \int_{\mathbb{T}} \cos t \, d\mu(t). \end{aligned} \quad (2.20)$$

Theorem 2.5. *Let V satisfy (1.1). Then we have the following estimates:*

$$\sum (1 - |z_j|) \leq -B_0 \leq C(d) \|V\|_{2/3} - \frac{\sigma(\mathbb{T})}{2\pi}. \quad (2.21)$$

and if $\operatorname{Im} V \geq 0$, then

$$\sum_{j=1}^N \operatorname{Im} \lambda_j \leq \operatorname{Tr} \operatorname{Im} V + C(d) \|V\|_{2/3}, \quad (2.22)$$

and if $V \geq 0$, then

$$\sum_{j=1}^N \sqrt{\lambda_j^2 - d^2} \leq \operatorname{Tr} V + C(d) \|V\|_{2/3}, \quad (2.23)$$

Remark.

Note that some of the results stated in Theorems 2.4 and 2.5 are new even for real-valued potentials, see Section 5.

3. DETERMINANTS

3.1. Properties of the Laplacian. One may diagonalize the discrete Laplacian, using the (unitary) Fourier transform $\Phi: \ell^2(\mathbb{Z}^d) \rightarrow L^2(\mathbb{S}^d)$, where $\mathbb{S} = \mathbb{R}/(2\pi\mathbb{Z})$. It is defined by

$$(\Phi f)(k) = \widehat{f}(k) = \frac{1}{(2\pi)^{\frac{d}{2}}} \sum_{n \in \mathbb{Z}^d} f_n e^{i(n,k)}, \quad \text{where } k = (k_j)_{j=1}^d \in \mathbb{S}^d.$$

Here (\cdot, \cdot) is the scalar product in \mathbb{R}^d . In the so-called momentum representation of the operator H , we have:

$$\Phi H \Phi^* = \widehat{\Delta} + \widehat{V}.$$

The Laplacian is transformed into the multiplication operator

$$(\widehat{\Delta} \widehat{f})(k) = h(k) \widehat{f}(k), \quad h(k) = \sum_1^d \cos k_j, \quad k \in \mathbb{S}^d,$$

and the potential V becomes a convolution operator

$$(\widehat{V} \widehat{f})(k) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{S}^d} \widehat{V}(k - k') \widehat{f}(k') dk',$$

where

$$\widehat{V}(k) = \frac{1}{(2\pi)^{\frac{d}{2}}} \sum_{n \in \mathbb{Z}^d} V_n e^{i(n,k)}, \quad V_n = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{S}^d} \widehat{V}(k) e^{-i(n,k)} dk.$$

3.2. Trace class operators. Here for the sake of completeness we give some standard facts from Operator Theory in Hilbert Spaces..

Let \mathcal{H} be a Hilbert space endowed with inner product (\cdot, \cdot) and norm $\|\cdot\|$. Let \mathcal{B}_1 be the set of all trace class operators on \mathcal{H} equipped with the trace norm $\|\cdot\|_{\mathcal{B}_1}$. Let us recall some well-known facts.

- Let $A, B \in \mathcal{B}$ and $AB, BA \in \mathcal{B}_1$. Then

$$\mathrm{Tr} AB = \mathrm{Tr} BA,$$

$$\det(I + AB) = \det(I + BA).$$

$$|\det(I + A)| \leq e^{\|A\|_{\mathcal{B}_1}}.$$

$$|\det(I + A) - \det(I + B)| \leq \|A - B\|_{\mathcal{B}_1} e^{1 + \|A\|_{\mathcal{B}_1} + \|B\|_{\mathcal{B}_1}}.$$

Moreover, $I + A$ is invertible if and only if $\det(I + A) \neq 0$.

- Suppose for a domain $\mathcal{D} \subset \mathbb{C}$, the function $\Omega(\cdot) - I : \Omega \rightarrow \mathcal{B}_1$ is analytic and invertible for any $z \in \mathcal{D}$. Then for $F(z) = \det \Omega(z)$ we have

$$F'(z) = F(z) \mathrm{Tr} (\Omega(z)^{-1} \Omega'(z)).$$

- Recall that for $K \in \mathcal{B}_1$ and $z \in \mathbb{C}$, the following identity holds true:

$$\det(I - zK) = \exp \left(- \int_0^z \mathrm{Tr} (K(1 - sK)^{-1}) ds \right)$$

(see e.g. [18], p.167, or [31], p.331).

3.3. Fredholm determinant. We recall here results from [20] about the asymptotics of the determinant $\mathcal{D}(\lambda) = \det(I + V R_0(\lambda))$ as $|\lambda| \rightarrow \infty$.

Lemma 3.1. *Let $V \in \ell^1(\mathbb{Z}^d)$. Then the determinant $\mathcal{D}(\lambda) = \det(I + V R_0(\lambda))$ is analytic in $\Lambda = \mathbb{C} \setminus [-d, d]$ and satisfies*

$$\mathcal{D}(\lambda) = 1 + O(1/\lambda) \quad \text{as } |\lambda| \rightarrow \infty,$$

uniformly in $\arg \lambda \in [0, 2\pi]$, and

$$\begin{aligned} \log \mathcal{D}(\lambda) &= - \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \mathrm{Tr} (V R_0(\lambda))^n, \\ \log \mathcal{D}(\lambda) &= - \sum_{n \geq 1} \frac{d_n}{n \lambda^n}, \quad d_n = \mathrm{Tr} (H^n - H_0^n), \end{aligned} \tag{3.1}$$

where the right-hand side is absolutely convergent for $|\lambda| > r_1$, $r_1 > 0$ being a sufficiently large constant. In particular,

$$\begin{aligned} d_1 &= \mathrm{Tr} V, \\ d_2 &= \mathrm{Tr} V^2, \\ d_3 &= \mathrm{Tr} (V^3 + 6d\tau^2 V), \\ d_4 &= \mathrm{Tr} (V^4 + 8d\tau^2 V^2 + 2\tau^2 (V_{\Delta}) V), \dots, \end{aligned} \tag{3.2}$$

where $V_{\Delta} = \sum_{i=1}^d (S_j V S_j^* + S_j^* V S_j)$ and $(S_j f)(n) = f(n + e_j)$ and $\tau = \frac{1}{2}$.

Recall the conformal mapping $\lambda(\cdot) : \mathbb{D} \rightarrow \Lambda$ is given by $\lambda(z) = \frac{d}{2}(z + \frac{1}{z})$, $|z| < 1$, and note that $|\lambda| \rightarrow \infty$ iff $z \rightarrow 0$. We consider the operator-valued function $Y(\lambda(z))$, $\lambda \in \Lambda$, defined by

$$Y(\lambda(z)) = V_2 X(\lambda(z)), \quad \text{where} \quad X(\lambda(z)) = |V_1|^{1/2} R_0(\lambda(z)) |V_1|^{1/2} V_1 |V_1|^{-1},$$

and where V_1 and V_2 are defined in (1.2).

Theorem 3.2. *Let V satisfy (1.1). Then the operator-valued function $Y(\lambda(z)) : \mathbb{D} \rightarrow \mathcal{B}_1$ is analytic in the unit disc \mathbb{D} and satisfies*

$$\|Y(\lambda(z))\|_{\mathcal{B}_1} \leq C(d) \|V\|_{2/3}, \quad \forall z \in \mathbb{D}.$$

Moreover, the function $D(z)$, $z \in \mathbb{D}$ belongs to \mathcal{H}_∞ and

$$\|D(\cdot)\|_{\mathcal{H}_\infty} \leq e^{C(d)\|V\|_{2/3}}. \quad (3.3)$$

Proof. The operator V_2 belongs to \mathcal{B}_2 and due to Theorem 7.3 (see Appendix 2) the operator-function $X(\cdot) : \Lambda \rightarrow \mathcal{B}_2$ satisfies the inequality

$$\|X(\lambda)\|_{\mathcal{B}_2} \leq C(d) \|V_1\|_2, \quad \forall \lambda \in \Lambda = \mathbb{C} \setminus [-d, d],$$

Thus the operator-valued function $Y(\lambda(z)) = V_2 X(\lambda(z)) : \mathbb{D} \rightarrow \mathcal{B}_1$ is of trace class. Moreover, the function $D(z)$, $z \in \mathbb{D}$, belongs to \mathcal{H}_∞ and due to (3.2) it satisfies (3.3). ■

The function $D(z) = \mathcal{D}(\lambda(z))$ is analytic in \mathbb{D} with the zeros given by

$$z_j = z(\lambda_j), \quad j = 1, 2, \dots, N,$$

where λ_j are zeros (counting with multiplicity) of $\mathcal{D}(\lambda)$ in $\Lambda = \mathbb{C} \setminus [-d, d]$, i.e., eigenvalues of H .

Lemma 3.3. *Let a potential $V \in \ell^1(\mathbb{Z}^d)$. Then $\log D(z)$ is analytic in \mathbb{D}_{r_0} defined by $\log D(0) = 0$, where r_0 is given by (2.4), and has the following Taylor series*

$$\log D(z) = -c_1 z - c_2 z^2 - c_3 z^3 - c_4 z^4 - \dots, \quad \text{as } |z| < r_0, \quad (3.4)$$

and

$$c_1 = d_1 a, \quad c_2 = d_2 a^2, \quad c_3 = d_3 a^3 - c_1, \quad c_4 = d_4 a^4 - c_2, \dots \quad (3.5)$$

where $a = \frac{2}{d}$ and the coefficients d_j are given by (3.2).

Proof. We have

$$\frac{1}{\lambda} = \frac{az}{1+z^2} = a(z - z^3 + O(z^5)), \quad \frac{1}{\lambda^2} = a^2(z^2 - z^4 + O(z^6)), \quad \frac{1}{\lambda^3} = a^3 z^3 + O(z^5)$$

as $z \rightarrow 0$. Substituting these asymptotics into (3.1) we obtain (3.4) and (3.5). ■

4. PROOF OF THE MAIN RESULTS

We are ready to prove main results.

Proof of Theorem 2.1. Let V satisfy (1.1). Then by Theorem 3.2, the determinant $D(z)$, $z \in \mathbb{D}$, is analytic and $D \in \mathcal{H}_\infty$. Moreover, Lemma 3.1 gives (2.5) and Lemma 3.3 gives (2.6)-(2.8). ■

Proof of Theorem 2.2. Due to Theorem 2.1 the determinant $D(z)$ is analytic in \mathbb{D} . Then Theorem 6.2 (see Appendix 1) yields

$$\mathcal{Z}_D := \sum_{j=1}^{\infty} (1 - |z_j|) < \infty$$

and the Blaschke product $B(z)$ given by

$$B(z) = \prod_{j=1}^N \frac{|z_j|}{z_j} \frac{z_j - z}{1 - \bar{z}_j z}, \quad z \in \mathbb{D},$$

converges absolutely for $\{|z| < 1\}$. We have $D(z) = B(z)D_B(z)$, where D_B is analytic in the unit disc \mathbb{D} and has no zeros in \mathbb{D} . Thus we have proved ii).

i) Lemma 6.1 gives (2.11) and (2.12).

iii) For small a sufficiently small z and for $t = z_j \in \mathbb{D}$ for some j we have the following identity:

$$\log \frac{|t|}{t} \frac{t - z}{1 - \bar{t}z} = \log |t| + \log \left(1 - \frac{z}{t}\right) - \log(1 - \bar{t}z) = \log |t| - \sum_{n \geq 1} \left(\frac{1}{t^n} - \bar{t}^n\right) \frac{z^n}{n}.$$

Besides,

$$|1 - |t|^n| \leq n|1 - |t||,$$

$$|t^{-n} - \bar{t}^n| \leq |1 - t^n| + |1 - t^{-n}| \leq |1 - t^n| \left(1 + \frac{1}{|t|^n}\right) \leq |1 - |t|^n| \frac{2}{r_0^n} \leq |1 - |t|| \frac{2n}{r_0^n},$$

where $r_0 = \inf |z_j| > 0$. This yields

$$\begin{aligned} \log B(z) &= \sum_{j=1}^N \log \frac{|z_j|}{z_j} \frac{z_j - z}{1 - \bar{z}_j z} = \sum_{j=1}^N \left(\log |z_j| + \log \left(1 - (z/z_j)\right) - \log(1 - \bar{z}_j z) \right) \\ &= \sum_{j=1}^N \log |z_j| - \sum_{n=1}^{\infty} \sum_{j=1}^N \left(\frac{1}{z_j^n} - \bar{z}_j^n \right) \frac{z^n}{n} = \log B(0) - b(z), \\ b(z) &= \sum_{n=1}^{\infty} \sum_{j=1}^N \left(\frac{1}{z_j^n} - \bar{z}_j^n \right) \frac{z^n}{n} = \sum_{n=1}^{\infty} z^n B_n, \quad B_n = \frac{1}{n} \sum_{j=1}^N \left(\frac{1}{z_j^n} - \bar{z}_j^n \right), \end{aligned} \tag{4.1}$$

where the function b is analytic in the disk $\{|z| < \frac{r_0}{2}\}$ and B_n satisfy

$$|B_n| \leq \frac{1}{n} \sum_{j=1}^N \left| \frac{1}{z_j^n} - \bar{z}_j^n \right| \leq \frac{2}{r_0^n} \sum_{j=1}^N |1 - |z_j|| = \frac{2}{r_0^n} \mathcal{Z}_D,$$

where $\mathcal{Z}_D = \sum_{j=1}^{\infty} (1 - |z_j|)$. Thus

$$|b(z)| \leq \sum_{n=1}^{\infty} |B_n| |z|^n \leq 2\mathcal{Z}_D \sum_{n=1}^{\infty} \frac{|z|^n}{r_0^n} = \frac{2\mathcal{Z}_D}{1 - \frac{|z|}{r_0}}.$$

■

Proof of Theorem 2.3.

i) Theorem 2.1 implies $D \in \mathcal{H}_{\infty}$. Therefore the canonical representation (2.14) follows from Lemma 6.3.

ii) The relation (6.3) gives the proof of ii). ■

Proof of Theorem 2.4. By using Lemma 6.4, (2.5)-(2.7) and Proposition 2.2 we obtain identities (2.16)-(2.19).

We have the following identities for $z \in \mathbb{D}$ and $\zeta = \frac{\lambda}{d} \in \Lambda_1$:

$$2\zeta = z + \frac{1}{z}, \quad z = \zeta - \sqrt{\zeta^2 - 1}, \quad z - \frac{1}{z} = -2\sqrt{\zeta^2 - 1}. \quad (4.2)$$

These identities yield

$$\begin{aligned} \bar{z} - \frac{1}{z} &= z + \bar{z} - 2\zeta = 2\operatorname{Re} z - 2\zeta, \\ \bar{z} - \frac{1}{z} &= \bar{z} - z - 2\sqrt{\zeta^2 - 1} = -2i\operatorname{Im} z - 2\sqrt{\zeta^2 - 1}. \end{aligned} \quad (4.3)$$

Then we get

$$\begin{aligned} \frac{2}{d} \operatorname{Tr} \operatorname{Im} V + \operatorname{Im} \sum_{j=1}^N \left(\bar{z}_j - \frac{1}{z_j} \right) &= \frac{1}{\pi} \int_{\mathbb{T}} \sin t \, d\mu(t), \\ \operatorname{Im} \left(\bar{z}_j - \frac{1}{z_j} \right) &= -2\operatorname{Im} \zeta_j = -\frac{2}{d} \operatorname{Im} \lambda_j, \\ \operatorname{Re} \left(\bar{z}_j - \frac{1}{z_j} \right) &= 2\operatorname{Re}(z_j - \zeta_j) = -2\operatorname{Re} \sqrt{\zeta_j^2 - 1} \end{aligned}$$

and thus

$$\begin{aligned} \sum_{j=1}^N \operatorname{Im} \lambda_j &= \operatorname{Tr} \operatorname{Im} V - \frac{d}{2\pi} \int_{\mathbb{T}} \sin t \, d\mu(t), \\ \sum_{j=1}^N \operatorname{Re} \sqrt{\lambda_j^2 - d^2} &= \operatorname{Tr} \operatorname{Re} V + \frac{d}{2\pi} \int_{\mathbb{T}} \cos t \, d\mu(t), \end{aligned}$$

■

Proof of Theorem 2.5. The simple inequality $1 - x \leq -\log x$ for $\forall x \in (0, 1]$, implies $-B_0 = -B(0) = -\sum \log |z_j| \geq \sum (1 - |z_j|)$. Then substituting the last estimate and the estimate (2.5) into the first trace formula (2.16) we obtain (2.21).

In order to determine the next two estimates we use the trace formula (2.16). Let $\text{Im } V \geq 0$. Then $\text{Im } \lambda_j \geq 0$ and the estimates (2.5) and (2.21) and the second trace formula (2.20) imply

$$\begin{aligned} \sum_{j=1}^N \text{Im } \lambda_j - \text{Tr } \text{Im } V &= -\frac{d}{2\pi} \int_{\mathbb{T}} \sin t \, d\mu(t) \\ &\leq \frac{d}{2\pi} \int_{\mathbb{T}} (C\|V\|_{\frac{2}{3}} dt + d\sigma(t)) \leq C(d)\|V\|_{\frac{2}{3}}, \end{aligned}$$

which yields (2.22). Similar arguments give (2.23). ■

5. SCHRÖDINGER OPERATORS WITH REAL POTENTIALS

Consider Schrödinger operators $H = -\Delta + V$, where the potential V is real and satisfies condition (1.1). The spectrum of H has the form

$$\sigma(H) = \sigma_{ac}(H) \cup \sigma_{sc}(H) \cup \sigma_p(H) \cup \sigma_{dis}(H), \quad \sigma_{ac}(H) = [-d, d],$$

where

$$\sigma_p(H) \subset [-d, d], \quad \sigma_{dis}(H) \subset \mathbb{R} \setminus [-d, d].$$

Note that each eigenvalues of H has a finite multiplicity.

5.1. Discrete spectrum. The discrete eigenvalues of the operator H are real and belong to the set $\mathbb{R} \setminus [-d, d]$. Let they are labeled by

$$\cdots \leq \lambda_{-2} \leq \lambda_{-1} < -d < d < \lambda_1 \leq \lambda_2 \leq \cdots$$

The corresponding point from $z_j \in \mathbb{D}$ is real and satisfy

$$\lambda_j = \frac{d}{2} \left(z_j + \frac{1}{z_j} \right), \quad j \in \mathbb{Z} \setminus \{0\}.$$

Moreover, we have the identity

$$\sqrt{\lambda^2 - d^2} = \frac{d}{2} \left(z - \frac{1}{z} \right)$$

for all $\lambda \in \Lambda$ and $z \in \mathbb{D}$. If λ is the eigenvalue of H , then we have the identity

$$\begin{aligned} \frac{d}{2} \left(\frac{1}{z} - z \right) &= -|\lambda^2 - d^2|^{\frac{1}{2}} \quad \text{if } \lambda < -d, \\ \frac{d}{2} \left(\frac{1}{z} - z \right) &= |\lambda^2 - d^2|^{\frac{1}{2}} \quad \text{if } \lambda > d. \end{aligned} \tag{5.1}$$

The next result follows immediately from Theorem 2.4.

Theorem 5.1. (The trace formulas.) *Let a real potential V satisfy (1.1). Then there is infinite number of trace formulae*

$$0 \leq \frac{\sigma(\mathbb{T})}{2\pi} - B_0 = \frac{1}{2\pi} \int_{\mathbb{T}} \log |D(e^{it})| dt \leq C(d, p) \|V\|_q, \tag{5.2}$$

$$- \text{Tr } V + \sum_{j=1}^N |\lambda_j^2 - d^2|^{\frac{1}{2}} \text{sign } \lambda_j = \frac{d}{2\pi} \int_{\mathbb{T}} e^{-it} d\mu(t),$$

$$-\operatorname{Tr} V^2 + \sum_{j=1}^N |\lambda_j| |\lambda_j^2 - d^2|^{\frac{1}{2}} = \frac{d^2}{4\pi} \int_{\mathbb{T}} e^{-i2t} d\mu(t), \quad \dots$$

Proof. The eigenvalue of H , then we have the identity

$$z = \frac{1}{d} \left(\lambda \pm \sqrt{\lambda^2 - d^2} \right).$$

■

Remark. 1) We consider the case (5.2). If $\lambda > d$, then we have $z \in (0, 1)$ and then

$$\begin{aligned} 1 - z &= 1 - \frac{1}{d} \left(\lambda - \sqrt{\lambda^2 - d^2} \right) = \frac{1}{d} \left(d - \lambda + \sqrt{\lambda^2 - d^2} \right) \\ &= \frac{\sqrt{\lambda - d}}{d} \left(\sqrt{\lambda + d} - \sqrt{\lambda - d} \right) = \frac{2\sqrt{\lambda - d}}{(\sqrt{\lambda + d} + \sqrt{\lambda - d})} \geq \frac{\sqrt{\lambda - d}}{\sqrt{\lambda + d}}. \end{aligned}$$

This yields

$$\sum_{\lambda_j > d} \frac{\sqrt{\lambda_j - d}}{\sqrt{\lambda_j + d}} + \sum_{\lambda_j < -d} \frac{\sqrt{-\lambda_j - d}}{\sqrt{-\lambda_j + d}} = \sum_{\lambda_j} \frac{\sqrt{|\lambda_j| - d}}{\sqrt{|\lambda_j| + d}} \leq C_d \|V\|_{2/3}.$$

Corollary 5.2. *Let a potential V be real and satisfy (1.1). Then the following estimates hold true:*

$$\sum_{j=1}^N |\lambda_j| |\lambda_j^2 - d^2|^{\frac{1}{2}} \leq \operatorname{Tr} V^2 + \frac{d^2}{4\pi} C(d, p) \|V\|_{2/3}.$$

6. APPENDIX, HARDY SPACES

6.1. Analytic functions. We recall the basic facts about the Blaschke product (see pages 53-55 in [17]) of zeros $\{z_n\}$. The subharmonic function $v(z)$ on Ω has a harmonic majorant if there is a harmonic function $U(z)$ such that $v(z) \leq U(z)$ throughout Ω .

We need the following well-known results, see e.g. Sect. 2 from [17].

Lemma 6.1. *Let $\{z_j\}$ be a sequence of points in $\mathbb{D} \setminus \{0\}$ such that $\sum (1 - |z_j|) < \infty$ and let $m \geq 0$ be an integer. Then the Blaschke product*

$$B(z) = z^m \prod_{z_j \neq 0} \frac{|z_j|}{z_j} \left(\frac{z_j - z}{1 - \bar{z}_j z} \right),$$

converges in \mathbb{D} . Moreover, the function B is in \mathcal{H}_∞ and zeros of B are precisely the points z_j , according to the multiplicity. Moreover,

$$\begin{aligned} |B(z)| &\leq 1 \quad \forall z \in \mathbb{D}, \\ \lim_{r \rightarrow 1} |B(re^{i\vartheta})| &= |B(e^{i\vartheta})| = 1 \quad \text{almost everywhere, } \vartheta \in \mathbb{T}, \\ \lim_{r \rightarrow 1} \int_0^{2\pi} \log |B(re^{i\vartheta})| d\vartheta &= 0. \end{aligned} \tag{6.1}$$

Let us recall a well-known result concerning analytic functions in the unit disc, e.g., see Koosis page 67 in [22].

Theorem 6.2. *Let f be analytic in the unit disc \mathbb{D} and let $z_j \neq 0, j = 1, 2, \dots, N \leq \infty$ be its zeros labeled by*

$$0 < |z_1| \leq \dots \leq |z_j| \leq |z_{j+1}| \leq |z_{j+2}| \leq \dots$$

Suppose that f satisfies the condition

$$\sup_{r \in (0,1)} \int_0^{2\pi} \log |f(re^{i\vartheta})| d\vartheta < \infty.$$

Then

$$\sum_{j=1}^N (1 - |z_j|) < \infty.$$

The Blaschke product $B(z)$ given by

$$B(z) = z^m \prod_{j=1}^N \frac{|z_j|}{z_j} \frac{(z_j - z)}{(1 - \bar{z}_j z)},$$

where m is the multiplicity of B at zero, converges absolutely for $\{|z| < 1\}$. Besides, $f_B(z) = f(z)/B(z)$ is analytic in the unit disc \mathbb{D} and has no zeros in \mathbb{D} .

Moreover, if $f(0) \neq 0$ and if $u(z)$ is the least harmonic majorant of $\log |f(z)|$, then

$$\sum (1 - |z_j|) < u(0) - \log |f(0)|.$$

We now consider the canonical representation (6.2) for a function $f \in \mathcal{H}_p, p > 0$ (see, [22], p. 76).

Lemma 6.3. *Let a function $f \in \mathcal{H}_p, p > 0$. Let B be its Blaschke product. Then there exists a singular measure $\sigma = \sigma_f \geq 0$ on $[-\pi, \pi]$ with*

$$\begin{aligned} f(z) &= B(z) e^{ic - K_\sigma(z)} e^{K_f(z)}, \\ K_\sigma(z) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} d\sigma(t), \\ K_f(z) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} \log |f(e^{it})| dt, \end{aligned} \tag{6.2}$$

for all $|z| < 1$, where c is real constant and $\log |f(e^{it})| \in L^1(-\pi, \pi)$.

We define the functions (after Beurling) in the disc by

$$\begin{aligned} f_{in}(z) &= B(z) e^{ic - K_\sigma(z)} && \text{the inner factor of } f, \\ f_{out}(z) &= e^{K_f(z)} && \text{the outer factor of } f, \end{aligned}$$

for $|z| < 1$. Note that we have $|f_{in}(z)| \leq 1$, since $d\sigma \geq 0$.

Thus $f_B(z) = \frac{f(z)}{B(z)}$ has no zeros in the disc \mathbb{D} and satisfies

$$\log f_B(z) = ic + \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} d\mu(t),$$

where the measure μ equals

$$d\mu(t) = \log |f(e^{it})| dt - d\sigma(t).$$

For a function f continuous on the disc $\overline{\mathbb{D}}$ we define the set of zeros of f lying on the boundary $\partial\mathbb{D}$ by

$$\mathfrak{S}_0(f) = \{z \in \mathbb{S} : f(z) = 0\}.$$

It is well known that the support of the singular measure $\sigma = \sigma_f$ satisfies

$$\text{supp } \sigma_f \subset \mathfrak{S}_0(f) = \{z \in \mathbb{S} : f(z) = 0\} \quad (6.3)$$

see for example, Hoffman [19], p. 70.

In the next statement we present trace formulae for a function $f \in \mathcal{H}_p, p > 0$.

Lemma 6.4. *Let $f \in \mathcal{H}_p, p > 0$ and $f(0) = 1$ and let B be its Blaschke product. Let the functions $\log f$ and $F = \log f_B$ have the Taylor series in some small disc $\mathbb{D}_r, r > 0$ given by*

$$\begin{aligned} \log f(z) &= -f_1 z - f_2 z^2 - f_3 z^3 - \dots, \\ F = \log f_B(z) &= F_0 + F_1 z + F_2 z^2 + F_3 z^3 + \dots, \\ \log B(z) &= B_0 - B_1 z - B_2 z^2 - \dots, \quad \text{as } z \rightarrow 0, \\ F_0 &= -\log B(0) > 0, \quad F_n = B_n - f_n, \quad n \geq 1. \end{aligned} \quad (6.4)$$

Then the factorization (6.2) holds true and we have

$$c = 0, \quad F_0 = -\log B(0) = \frac{\mu(\mathbb{T})}{2\pi} \geq 0, \quad (6.5)$$

$$F_n = \frac{1}{\pi} \int_{-\pi}^{\pi} e^{-int} d\mu(t), \quad n = 1, 2, \dots, \quad (6.6)$$

where the measure $d\mu(t) = \log |f(e^{it})| dt - d\sigma(t)$.

Proof. Recall that the identity (6.2) gives $f(z) = B(z)e^{ic-K_\sigma(z)}e^{K_f(z)}$, then at $z = 0$ we obtain

$$1 = f(0) = B(0)e^{ic-K_\sigma(0)}e^{K_f(0)}.$$

Since $B(0), K_\sigma(0), K_f(0)$ and c are real we obtain $c = 0$. Moreover, the inequality (6.1) implies $F_0 \geq 0$.

In order to show (6.6) we need the asymptotics of the Schwatz integral

$$f(z) = B(z)f_B(z), \quad F(z) = \log f_B(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} d\mu(t), \quad (6.7)$$

as $z \rightarrow 0$. The following identity holds true

$$\frac{e^{it} + z}{e^{it} - z} = 1 + \frac{2ze^{-it}}{1 - ze^{-it}} = 1 + 2 \sum_{n \geq 1} (ze^{-it})^n = 1 + 2(ze^{-it}) + 2(ze^{-it})^2 + \dots \quad (6.8)$$

Thus (6.7), (6.8) yield the Taylor series at $z = 0$:

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} d\mu(t) = \frac{\mu(\mathbb{T})}{2\pi} + \mu_1 z + \mu_2 z^2 + \mu_3 z^3 + \mu_4 z^4 + \dots \quad \text{as } z \rightarrow 0, \quad (6.9)$$

where

$$\mu(\mathbb{T}) = \int_0^{2\pi} d\mu(t), \quad \mu_n = \frac{1}{\pi} \int_0^{2\pi} e^{-in\vartheta} d\mu(t), \quad n \in \mathbb{Z}.$$

Thus comparing (6.4) and (6.9) we obtain

$$-\log B(0) = \frac{\mu(\mathbb{T})}{2\pi} \geq 0, \quad F_n = \mu_n \quad \forall n \geq 1.$$

■

7. APPENDIX, ESTIMATES INVOLVING BESSEL'S FUNCTIONS

In order complete the proof of Theorem 3.2 we need some uniform estimates for the Bessel functions $J_m, m \in \mathbb{Z}$ with respect to m for which we their integral representation

$$J_m(t) = \frac{1}{2\pi} \int_0^{2\pi} e^{-imk - it \sin k} dk = \frac{i^m}{2\pi} \int_0^{2\pi} e^{-imk + it \cos k} dk. \quad (7.1)$$

Note that for all $(t, m) \in \mathbb{R} \times \mathbb{Z}$:

$$J_{-m}(t) = J_m(t) \quad \text{and} \quad J_m(-t) = (-1)^m J_m(t). \quad (7.2)$$

Our estimates are based on the following three asymptotic formulae, see [35], Ch IV, § 2. Let

$$\xi = \frac{m}{t}.$$

Then for a fixed $\varepsilon, 0 < \varepsilon < 1$ we have

1. if $\xi > 1 + \varepsilon$ then

$$J_m(t) = \frac{1}{2} \sqrt{\frac{2}{\pi t}} \frac{1}{(\xi^2 - 1)^{1/4}} e^{-t(\xi \ln(\xi + \sqrt{\xi^2 - 1}) - \sqrt{\xi^2 - 1})} \left(1 + O\left(\frac{1}{m}\right)\right). \quad (7.3)$$

Therefore if $\xi > 1 + \varepsilon$ then this formula implies the uniform with respect to m exponential decay of the Bessel function in t .

2. if $\xi < 1 - \varepsilon$ then

$$J_m(t) = \frac{1}{2} \sqrt{\frac{2}{\pi t}} \frac{1}{(\xi^2 - 1)^{1/4}} \left(e^{-i\pi/4 + it(-\xi \arccos \xi + \sqrt{1 - \xi^2})} + e^{i\pi/4 + it(\xi \arccos \xi - \sqrt{1 - \xi^2})} \right) \left(1 + O\left(\frac{1}{t}\right)\right). \quad (7.4)$$

In this case the Bessel function oscillates as $t \rightarrow \infty$ and obviously the latter formula implies the uniform with respect to m estimate

$$|J_m(t)| \leq C t^{-1/2}, \quad C = C(\varepsilon).$$

3. We now consider the third case $1 - \varepsilon \leq \xi \leq 1 + \varepsilon$ which is more difficult.

Lemma 7.1. *If $1 - \varepsilon \leq \xi \leq 1 + \varepsilon, \varepsilon > 0$, then there is a constant $C = C(\varepsilon)$ such that*

$$J_m(t) \leq C t^{-1/4} \left(|t|^{1/3} + |m - t| \right)^{-1/4}, \quad \forall m, t, \quad |m - t| < \varepsilon t. \quad (7.5)$$

Proof. If $1 - \varepsilon \leq \xi \leq 1 + \varepsilon$, then (see [35], Ch IV, § 2)

$$J_m(t) = \frac{v(t^{2/3}\tau(\xi))}{t^{1/3}} \left(c_0(\xi) + O\left(\frac{1}{t}\right) \right) + \frac{v'(t^{2/3}\tau(\xi))}{t^{4/3}} \left(d_0(\xi) + O\left(\frac{1}{t}\right) \right), \quad (7.6)$$

where v is the Airy function and

$$\tau^{3/2}(\xi) = \xi \ln \left(\xi + \sqrt{\xi^2 - 1} \right) - \sqrt{\xi^2 - 1}$$

and therefore

$$\tau(\xi) = 2^{1/3}(\xi - 1) + O((\xi - 1)^2), \quad \text{as } \xi \rightarrow 1. \quad (7.7)$$

Besides the functions $c_0(\xi)$ and $d_0(\xi)$ are bounded with respect to ξ and, for example,

$$c_0(\xi) = \sqrt{\frac{2}{\pi}} \left(\frac{\tau(\xi)}{\xi^2 - 1} \right)^{1/4},$$

(see [28] formulae (10.06), (10.07))

In what follows all the constants depend on ε , $0 < \varepsilon < 1$, but not on m and t . Due (7.7) there are constants c and C such that

$$c(\varepsilon)(1 + |y|)^{1/4} \leq \left(t^{1/3} + |m - t| \right)^{1/4} t^{-1/12} \leq C(\varepsilon)(1 + |y|)^{1/4}, \quad (7.8)$$

where $y = t^{2/3} \tau(\xi)$. Moreover, since $|\xi - 1| = \left| \frac{m}{t} - 1 \right| \leq \varepsilon$ we also have

$$\left(t^{1/3} + |m - t| \right)^{1/4} t^{-1/12} \leq C(\varepsilon) t^{1/6}. \quad (7.9)$$

Applying estimates for the Airy functions in (7.6)

$$|v(y)| \leq C(1 + |y|)^{-1/4}, \quad |v'(y)| \leq C(1 + |y|)^{1/4}$$

and using (7.8), (7.9) we find that if $|t| \geq 1$

$$\begin{aligned} |J_m(t)| &\leq C \left(\frac{1}{(1 + |y|)^{1/4} |t|^{1/3}} + \frac{(1 + |y|)^{1/4}}{|t|^{4/3}} \right) \\ &\leq C \left(\frac{t^{1/12}}{t^{1/3} \left(t^{1/3} + |m - t| \right)^{1/4}} + \frac{\left(t^{1/3} + |m - t| \right)^{1/4}}{t^{4/3} t^{1/12}} \right) \\ &\leq C t^{-1/4} \left(|t|^{1/3} + |m - t| \right)^{-1/4}. \end{aligned}$$

The proof is complete. ■

Let us now consider the operator $e^{it\Delta}$, $t \in \mathbb{R}$. It is unitary on $L^2(\mathbb{S}^d)$ and its kernel is given by

$$(e^{it\Delta})(n - n') = \frac{1}{(2\pi)^d} \int_{\mathbb{S}^d} e^{-i(n - n', k) + i t h(k)} dk, \quad n, n' \in \mathbb{Z}^d. \quad (7.10)$$

where $h(k) = \sum_{j=1}^d \cos k_j$, $k = (k_j)_{j=1}^d \in \mathbb{S}^d$.

Lemma 7.2. *Let $n = (n_j)_{j=1}^d \in \mathbb{Z}^d$, $d \geq 1$. Then*

$$(e^{it\Delta})(n) = i^{-|n|} \prod_{j=1}^d J_{n_j}(t), \quad (n, t) \in \mathbb{Z}^d \times \mathbb{R}, \quad (7.11)$$

where $|n| = |n_1| + \dots + |n_d|$. Moreover, the following estimates are satisfied:

$$|(e^{it\Delta})(n)| \leq C_1 |t|^{-\frac{d}{3}}, \quad t \geq 1, \quad (7.12)$$

$$\int_1^\infty |J_m(t)|^d dt < C_2, \quad \text{if } m \in \mathbb{Z}, d \geq 3, \quad (7.13)$$

for all $(t, n) \in \mathbb{R} \times \mathbb{Z}^d$ and some constants $C_1 = C_1(d)$ and $C_2 = C_2(d)$.

Proof. Let $d = 1$ and $h(k) = \cos k, k \in \mathbb{T}$. Then using (7.10) and (7.1) we obtain

$$(e^{it\Delta})(n) = \frac{1}{(2\pi)} \int_{\mathbb{T}} e^{-ink+it \cos k} dk = i^{-n} J_n(t), \quad \forall (n, t) \in \mathbb{Z} \times \mathbb{R}.$$

which yields (7.11) for $d = 1$. Due to the separation of variables we also obtain (7.11) for any $d \geq 1$.

In view of (7.2) it is enough to consider the case $n_j \geq 0$. In order to obtain (7.12) it is enough to apply the inequalities (7.3), (7.4) and also (7.5) if in this inequality we ignore the term $|m - t|$.

If $d > 3$, then (7.12) implies (7.13). Let now $d = 3$. From (7.5) we obtain

$$\begin{aligned} \int_1^\infty |J_m(t)|^d dt &\leq C \left(\int_1^\infty |t|^{-3/2} dt + \int_1^\infty |t|^{-\frac{3}{4}} \left(|t|^{\frac{1}{3}} + |m - t| \right)^{-\frac{3}{4}} dt \right) \\ &\leq C/2 + C \int_1^\infty |t|^{-\frac{3}{4}} (1 + |m - t|)^{-\frac{3}{4}} dt \\ &\leq C/2 + C \left(\int_1^\infty |t|^{-3/2} dt \right)^{1/2} \left(\int_1^\infty (1 + |m - t|)^{-3/2} dt \right)^{1/2} < \infty. \end{aligned}$$

■

Theorem 7.3. *i) Let $d \geq 3$. Then for each $n \in \mathbb{Z}^d$ the following estimate holds true:*

$$\int_1^\infty |(e^{\pm it\Delta})(n)| dt \leq \beta, \quad (7.14)$$

where

$$\beta = \sup_{m \in \mathbb{Z}} \int_1^\infty |J_m(t)|^d dt < \infty. \quad (7.15)$$

ii) Let a function $q \in \ell^2(\mathbb{Z}^d)$ and let $X(\lambda) = qR_0(\lambda)q, \lambda \in \Lambda$. Then the operator-valued function $X : \Lambda \rightarrow \mathcal{B}_2$ is analytic and satisfies

$$\sup_{\lambda \in \Lambda} \|X(\lambda)\|_{\mathcal{B}_2} \leq (1 + \beta) \|q\|_2^2, \quad (7.16)$$

Proof.

i) Note that (7.13) gives (7.15). Due to (7.2) it is sufficient to show (7.14) for $n \in (\mathbb{Z}_+)^d$. Using (7.11) and (7.15), we obtain

$$\int_1^\infty |(e^{it\Delta})(n)| dt = \int_1^\infty \prod_1^d |J_{n_j}(t)| dt \leq \prod_1^d \left(\int_1^\infty |J_{n_j}(t)|^d dt \right)^{1/d} \leq \beta,$$

which yields (7.14).

ii) Consider the case \mathbb{C}_- , the proof for \mathbb{C}_+ is similar. We have the standard representation of the free resolvent $R_0(\lambda)$ in the lower half-plane \mathbb{C}_- given by

$$R_0(\lambda) = -i \int_0^\infty e^{it(\Delta-\lambda)} dt = R_{01}(\lambda) + R_{02}(\lambda),$$

$$R_{01}(\lambda) = -i \int_0^1 e^{it(\Delta-\lambda)} dt, \quad R_{02}(\lambda) = -i \int_1^\infty e^{it(\Delta-\lambda)} dt,$$

for all $\lambda \in \mathbb{C}_-$. Here the operator valued-function $R_{01}(\lambda)$ has analytic extension from \mathbb{C}_- into the whole complex plane \mathbb{C} and satisfies

$$\|R_{01}(\lambda)\| \leq 1, \quad \|qR_{01}(\lambda)q\|_{\mathcal{B}_2} \leq \|q\|_2^2 \quad \forall \lambda \in \mathbb{C}_-,$$

Let $R_{02}(n' - n, \lambda)$ be the kernel of the operator $R_{02}(\lambda)$. We have the identity

$$R_{02}(m, \lambda) = -i \int_1^\infty (e^{it(\Delta-\lambda)})(m) dt, \quad m = n' - n.$$

Then the estimate (7.14) gives

$$|R_{02}(m, \lambda)| \leq \int_1^\infty |(e^{it\Delta})(m)| dt \leq \beta,$$

which yields

$$\|qR_{02}(\lambda)q\|_{\mathcal{B}_2}^2 = \sum_{n, n' \in \mathbb{Z}^d} |q(n)|^2 |R_{02}(n - n', \lambda)|^2 |q(n)|^2 \leq \sum_{n, n' \in \mathbb{Z}^d} |q(n)|^2 \beta^2 |q(n)|^2 = \beta^2 \|q\|_2^4,$$

and summing results for R_{01} and R_{02} we obtain (7.16). ■

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